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Partition sums and entropy bounds in weakly coupled CFT

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Abstract: We use the partition functions on $S^1 \times S^n$ of various conformal field theories in four and six dimensions in the limit of vanishing coupling to study the high temperature thermodynamics. Certain modular properties exhibited by the partition functions help to determine the finite volume corrections, which play a role in the discussion of entropy bounds.

Keywords: Field Theories in Higher Dimensions, Conformal Field Models in String Theory, AdS-CFT Correspondance.

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Contents

1. Introduction

Twenty years ago, Bekenstein proposed [1] that the entropy of a complete physical system in three spatial dimensions whose total energy is E and which fits in a sphere of radius R, is necessarily bounded from above,

$$S \leq 2\pi ER\,. \tag{1.1}$$

The motivation for the bound came from studying the consequences of the generalized second law (GSL) of thermodynamics (i.e. thermodynamics in the presence of black holes). When an entropy-bearing object is dropped into a black hole, the GSL appears to be violated unless the infalling object satisfies the bound (1.1). A similar bound should hold by the same arguments in other spacetime dimensions.

Over the years, various objections to the bound (1.1) have been raised (see [2, 3] for recent reviews). In addition to the fact that it is not clear how to define E and R in a general non-spherically symmetric spacetime, it is now also not clear that the bound (1.1) is necessary for the validity of the GSL [2].

An interesting feature of the bound (1.1) is that it does not involve the gravitational coupling constant G_N, and thus should remain valid in the limit $G_N \to 0$. In other words it should be a property of the non-gravitational dynamics in a fixed background spacetime, and can be checked directly in model systems. Many such checks have been performed [3], but the situation in some of these cases and in general seems unclear.

A natural class of theories in which to study the bound (1.1) is quantum field theories in D dimensions, and in particular the conformal field theories (CFT's) to which they flow in the extreme UV and IR limits. The latter problem was discussed in an interesting recent paper by E. Verlinde [4], who argued that the entropy of a general D-dimensional CFT on

$$\mathbb{R} \times S^{D-1}, \tag{1.2}$$

is related to the energy via

$$S = \frac{2\pi R}{D-1}\sqrt{E_C(2E - E_C)}\,, \tag{1.3}$$

where R is the radius of S^{D-1}, and E_C is the sub-extensive part of the energy E. It is defined through

$$E_C \equiv DE - (D-1)TS\,, \tag{1.4}$$

and is parametrically smaller than the energy for high temperature, $TR \gg 1$. The evidence supporting (1.3) is twofold:

1. In $D = 2$, (1.3) reduces to Cardy's asymptotic entropy formula [5]

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}\,, \tag{1.5}$$

   which is valid[1] for general CFT's in $D = 2$.

2. According to the AdS/CFT correspondence [6] a large class of CFT's are described by a holographic dual. For such theories, (1.3) is valid in the strong coupling regime where the dual theory reduces to supergravity [4], for all energies above the Hawking-Page phase transition.

A generalization like (1.3) of the Cardy formula (1.5) that is universally true would be very interesting, both because it implies that the relation (1.1) is always satisfied in the region of validity of (1.3), and for other applications. Since it seems to hold for strongly coupled CFT's with AdS duals [4], it is natural to ask whether it also holds in the opposite limit of vanishing coupling. This question is analyzed in this paper; unfortunately the answer is negative. We arrive at this conclusion by computing the partition functions of different CFT's and analyzing their high temperature

[1]More precisely, (1.5) is valid in unitary CFT's with $c \gg 1$ and a gap in the spectrum of scaling dimensions, for $L_0 \gg c/24$.

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thermodynamics. The techniques we use are applicable for general dimension D, but we study in detail the cases of $D = 4$ and $D = 6$.

Before turning to the detailed calculations, we summarize some of the qualitative features that are common to the different examples we study.

1. The high temperature expansion of the free energy has the form
$$-FR = a_D(2\pi RT)^D + a_{D-2}(2\pi RT)^{D-2} + \cdots + a_0(2\pi RT)^0 + \mathcal{O}(e^{-(2\pi)^2 RT}) \,. \quad (1.6)$$
The perturbative part of the free energy is thus a finite polynomial *with no inverse powers in the temperature.* The non-perturbative corrections are of a specific form, namely a power series in $e^{-(2\pi)^2 TR}$.

2. The coefficients in the expansion (1.6) satisfy
$$\sum_{k=0}^{D/2} (-1)^k (2k-1) a_{2k} = 0 \,. \quad (1.7)$$
In two dimensions this gives $a_2 + a_0 = 0$ — the leading term of $\mathcal{O}((RT)^2)$ is identical (up to a sign) to the term independent of the temperature. This is indeed a well-known feature of (unitary) CFT's in $D = 2$. The sum rule (1.7) generalizes this to higher dimensions. The polynomial part of the free energy is characterized by $D/2$ coefficients and one constraint, for a net $D/2 - 1$ parameters, whose values in the different examples are given below in (4.5) and (4.11).

3. It is surprisingly useful to study the transformation under exchange of the radii $\beta/2\pi$ and R of the S^1 and the $S^{(D-1)}$, generalizing a modular transformation in $D = 2$. In higher dimensions this is generally not a full-fledged symmetry; nevertheless free field partition functions transform in a simple way.

The paper is organized as follows. In section 2 we compute the partition functions for various conformal field theories. Section 3 contains an analysis of the high temperature expansions of certain building blocks of these partition functions. The result is used in section 4 to find the high temperature expansions of the free energies of these CFT's. In section 5 we discuss the implications of our results to entropy bounds.

2. The partition functions of free CFTs

To study the thermodynamics of CFT on (1.2) one is interested in the spectrum of the Hamiltonian on S^{D-1}. The coordinate change $\tau = R\ln(\rho/R)$ maps the line element
$$ds^2 = d\tau^2 + R^2 d\Omega_{D-1}^2 \,, \quad (2.1)$$
on (1.2) (with euclidean time) to
$$ds^2 = \frac{R^2}{\rho^2}(d\rho^2 + \rho^2 d\Omega_{D-1}^2) \,, \quad (2.2)$$

and makes it manifest that $\mathbb{R}\times S^{D-1}$ is conformally equivalent to the euclidian space $\mathbb{R}^D$. Under this map the generator of time translations on (2.1) is related to the generator of scale transformations on (2.2) as

$$R\frac{\partial}{\partial\tau} = \rho\frac{\partial}{\partial\rho}\,. \tag{2.3}$$

The energy E of a state on $\mathbb{R}\times S^{D-1}$ is therefore related to the scaling dimension of the corresponding field as

$$ER = \Delta\,. \tag{2.4}$$

The partition sum

$$Z = \mathrm{Tr}\, e^{-\beta H} \tag{2.5}$$

of the CFT on (1.2) is obtained by compactifying euclidean time τ (2.1) on a circle of circumference β, $\tau\sim\tau+\beta$. Via the correspondence (2.4) it can be reinterpreted as the generating functional of conformal dimensions on $\mathbb{R}^D$,

$$Z = \sum_\Delta q^\Delta\,, \tag{2.6}$$

with

$$q = e^{-\beta/R}\,. \tag{2.7}$$

To compute it one can either enumerate local operators in the CFT on $\mathbb{R}^D$, or perform the quadratic path integral directly by analyzing the appropriate wave equation on $S^1\times S^{D-1}$. In the free theory either approach is manageable. In the following we carry out each of these derivations in the simplest case. We note some group theory which helps to automate the computation for other fields.

2.1 Four dimensions

We begin by considering the conformally coupled scalar in $D=4$. The equation of motion is

$$[-\nabla^2 + \xi\mathcal{R}]\phi = 0\,,$$

where the conformal coupling in $D=4$ is $\xi = 1/6$ and the Ricci tensor of the spatial sphere is $\mathcal{R} = 6/R^2$. The kinetic term $-\nabla^2$ on $S^1\times S^3$ contributes the energy $-E^2$ and the centrifugal term $n(n+2)/R^2$ where n is the integer partial wave number on the S^3. Altogether this gives the spectrum

$$\Delta = ER = n+1\,,\qquad n = 0,1,\ldots \tag{2.8}$$

The degeneracy comes exclusively from the wave functions on the sphere S^3. The dimension of the spin $(n/2,n/2)$ representation of $\mathrm{SO}(4)\simeq\mathrm{SU}(2)\times\mathrm{SU}(2)$ corresponding to the n'th partial wave is $(n+1)^2$.

This type of computation can be repeated for other fields. However, it is generally awkward to keep track of the couplings to the background curvature for fields other than the scalar and those details will at any rate turn out unimportant; so we will not record them here (see e.g. [7] for further details).

We now turn to the alternative strategy of enumerating free operators on the flat background $\mathbb{R}^4$ [8], considering again the conformally coupled scalar. The field ϕ has dimension $\Delta = 1$. Its derivatives $\partial_\mu \phi$ have dimension $\Delta = 2$ and degeneracy 4 due to the indices μ. At the next level there are $5 \cdot 4/2 = 10$ operators $\partial_\mu \partial_\nu \phi$ with $\Delta = 3$. But now there is also a constraint $-\nabla^2 \phi = 0$ from the equation of motion (in flat space) so there is a net degeneracy 9. Proceeding similarly to all orders one recovers (2.8) with the expected degeneracy $(n+1)^2$.

It is instructive to carry out this sort of counting for the first few levels of all the CFTs we consider. However, it is clearly tedious and not particularly illuminating to work out the combinatorics at arbitrary level. It is simpler to infer the complete tower of SO(4) representations from the existence of a partial wave expansion. This reasoning can be justified as follows. At level $\Delta = n+1$ the field transforms in the $(n/2, n/2)$ of SO(4). Acting with the derivative ∂_μ yields

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{n}{2}, \frac{n}{2}\right)\Big|_{\rm sym} = \left(\frac{n+1}{2}, \frac{n+1}{2}\right) \oplus \left(\frac{n-1}{2}, \frac{n-1}{2}\right). \tag{2.9}$$

The restriction to the symmetric part reflects symmetrization of multiple derivatives. Now, the field equation transforms as $((n-1)/2, (n-1)/2)$ and removes the final term so we are left with $((n+1)/2, (n+1)/2)$, the appropriate representation content at one level higher. In this way we find a full tower of operators. This computation generalizes to fields with spin.

Let us consider the Maxwell field. The field strength $F_{\mu\nu}$ has dimension $\Delta = 2$ and there are 6 components. At the next level the operators $\partial_\lambda F_{\mu\nu}$ with $\Delta = 3$ have 24 components. The constraints imposed by the 4 field equations $\partial^\mu F_{\mu\nu} = 0$ and the 4 Bianchi identities $\partial^{\mu *} F_{\mu\nu} = 0$ yield net degeneracy of 16. At level $\Delta = 4$ similar computations give a degeneracy of 40. These degeneracies agree with those of two SO(4) towers of the form $(n/2, (n+2)/2) + ((n+2)/2, n/2)$. The precise representations were determined from the SO(4) content at low level. In the free field limit it is in fact consistent to consider the two helicities independently (the self-dual and the anti-self dual part of the field).

Repeating the exercise for fermions completes table 1.

field	Δ	degeneracy	SO(4) rep.
Scalar	$n+1$	$(n+1)^2 = 1, 4, 9, \ldots$	$(n/2, n/2)$
Weyl fermion	$n+3/2$	$2(n+1)(n+2) = 4, 12, 24, \ldots$	$2(n/2, (n+1)/2)$
Vector	$n+2$	$2(n+1)(n+3) = 6, 16, 30, \ldots$	$(n/2, (n+2)/2) + h.c.$

Table 1: Spectrum of free fields in four dimensions. The range of $n = 0, 1, \ldots$ A definite chirality was chosen for the Weyl fermion. Both chiralities were included for the vector field (Maxwell case).

Alternatively the results can be summarized concisely in terms of the partition functions (2.6):

$$
\begin{aligned}
Z_S^{(4)} &= \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{n+1}} \right)^{(n+1)^2}, \\
Z_W^{(4)} &= \prod_{n=0}^{\infty} \left(1+q^{n+\frac{1}{2}} \right)^{2n(n+1)}, \\
Z_V &= \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{n+1}} \right)^{2n(n+2)}.
\end{aligned} \tag{2.10}
$$

Note that for the fermion and the vector the range of the index n was shifted compared with the table without changing the partition function.

$N = 4$ SYM theory in $D = 4$ with gauge group U(1) has the field content: 6 scalars, 4 Weyl fermions, and one vector field. It therefore has the partition function

$$
Z_{N=4} = (Z_S^{(4)})^6 \, (Z_W^{(4)})^4 \, Z_V \,.
$$

Different gauge groups (with more gauge fields) can be incorporated in the free field limit by simply taking the appropriate power of the entire partition function.

2.2 Six dimensions

The analysis of the six dimensional case is quite similar to the discussion for $D = 4$ so we shall be brief. Let us consider the self-dual tensor which may be less familiar. It is described by the field strength $H_{\mu\nu\rho}$ with conformal dimension $\Delta = 3$. Taking the antisymmetry in each index into account this gives $6 \cdot 5 \cdot 4/3! = 20$ components in six dimensions. The duality condition ${}^*H = H$ plays the role of the field equation. It removes half the components for a net degeneracy of 10. At the next level there are the fields $\partial_\lambda H_{\mu\nu\rho}$ with $\Delta = 4$. There are $6 \cdot 10$ such components after the duality condition on H has been taken into account. The trace $\partial^\mu H_{\mu\nu\rho} = 0$ gives further 15 conditions for a net degeneracy of 45. Note that the equation $\partial^\mu \, {}^*H_{\mu\nu\rho} = 0$ gives nothing new. It is now clear that the tensor field generate a tower with SO(6) quantum numbers $[2, n, 0]$ with $n = 0, 1, \ldots$ (see e.g. [9] for further details on these representations). The conjugate tower $[0, n, 2]$ corresponds to a field with the opposite duality condition.

The spectrum of the free fields in six dimensions is shown in table 2.

The corresponding partition functions are

$$
\begin{aligned}
Z_S^{(6)} &= \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{n+1}} \right)^{n(n+1)^2(n+2)/12}, \\
Z_W^{(6)} &= \prod_{n=0}^{\infty} \left(1+q^{n+\frac{1}{2}} \right)^{(n-1)n(n+1)(n+2)/3},
\end{aligned}
$$

field	Δ	degeneracy	SO(6) rep.
Scalar	$n+2$	$\frac{(n+1)(n+2)^2(n+3)}{12} = 1, 6, 20 \ldots$	$[0, n, 0]$
Weyl fermion	$n+5/2$	$\frac{(n+1)(n+2)(n+3)(n+4)}{3} = 8, 40, 120, \ldots$	$[1, n, 0]$
Tensor	$n+3$	$\frac{(n+1)(n+2)(n+4)(n+5)}{4} = 10, 45, 126, \ldots$	$[2, n, 0]$

Table 2: Spectrum of free fields in six dimensions. The range of $n = 0, 1, \ldots$.A definite chirality was chosen for the Weyl fermion and the tensor.

$$Z_T = \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{n+1}} \right)^{(n-1)n(n+2)(n+3)/4} . \tag{2.11}$$

The $(2,0)$ field theory in $D = 6$ has the field content: 5 scalars, 2 Weyl fermions, and one tensor field. It therefore has partition function

$$Z_{(2,0)} = (Z_S^{(6)})^5 \; (Z_W^{(6)})^2 \; Z_T \, .$$

Again we can consider a theory with N independent species, described by taking the entire partition function to the appropriate power. This is relevant for describing the $(2,0)$ CFT corresponding to N fivebranes at a generic point on the moduli space of vacua (where the fivebranes are separated). Coincident fivebranes are described by a strongly interacting CFT to which the above discussion does not apply.

3. Analysis of partition functions

To construct the high temperature thermodynamics of the partition sums (2.10), (2.11), we start in this section by analyzing the properties of certain basic partition functions. In the following section the results are applied to study the specific partition functions found in section 2.

3.1 The bosons

The partition function

$$Z_B^{(d)} = \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{n+1}} \right)^{(n+1)^{d-2}} , \tag{3.1}$$

can be interpreted as a simple model for a bosonic field in $D = d$ dimensions. We assume that the parameter d is even and use the notation $q = e^{-2\pi\delta} = e^{-\beta/R}$ i.e. $\delta = (2\pi RT)^{-1}$.

In the following we generalize a computation by Cardy [8]. First, take the logarithm of the partition sum and expand

$$-\delta^{d/2} \frac{\partial}{\partial\delta} \ln Z_B^{(d)} = 2\pi \sum_{n=0}^{\infty} (n+1)^{d-1} \delta^{d/2} \sum_{k=1}^{\infty} e^{-2\pi k(n+1)\delta} .$$

Then use the representation

$$e^{-x} = \frac{1}{2\pi i}\int_C x^{-(s+d/2)}\Gamma\left(s+\frac{d}{2}\right)ds\,,$$

where the contour C is along the imaginary axis, with the real part of s large ($\mathrm{Re}\, s > d/2$). Finally arrive at the Mellin transform

$$-\delta^{d/2}\frac{\partial}{\partial\delta}\mathrm{ln}Z_B^{(d)} = \frac{1}{2\pi i}\int_C \delta^{-s}G_B^{(d)}(s)ds\,, \tag{3.2}$$

where

$$G_B^{(d)}(s) = (2\pi)^{-s+1-k/2}\Gamma\left(s+\frac{d}{2}\right)\zeta\left(s+\frac{d}{2}\right)\zeta\left(s+1-\frac{d}{2}\right).$$

The function $G_B^{(d)}$ is meromorphic with poles

$$G_B^{(d)}(s) \sim 2\pi\,\frac{|B_d|}{2d}\,\frac{1}{s-d/2}\,,\qquad s\sim\frac{d}{2}\,, \tag{3.3}$$

and

$$G_B^{(d)}(s) \sim 2\pi\,\frac{B_d}{2d}\,\frac{1}{s+d/2}\,,\qquad s\sim -\frac{d}{2}\,. \tag{3.4}$$

We used

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|\,,\qquad \zeta(1-2n) = -\frac{B_{2n}}{2n}\,,$$

for $n\in\mathbb{Z}_+$, and also $\zeta(0) = -1/2$. Here B_d are the Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ etc. For $d = 2$ there is an additional pole

$$G_B^{(2)}(s) \sim -\frac{1}{2s}\,,\qquad s\sim 0\,.$$

Integration around the poles in (3.2) yields an expression for the partition functions at small δ:

$$\mathrm{ln}Z_B^{(d)} \simeq 2\pi\frac{|B_d|}{2d}\left(\frac{1}{d-1}\delta^{-d+1} + (-1)^{d/2}\delta\right) + \frac{1}{2}\mathrm{ln}\delta\,\delta_{d,2}\,. \tag{3.5}$$

For reference we write out the first few partition functions

$$\begin{aligned}
\mathrm{ln}Z_B^{(2)} &\simeq 2\pi\frac{1}{24}(\delta^{-1}-\delta) + \frac{1}{2}\mathrm{ln}\delta\,,\\
\mathrm{ln}Z_B^{(4)} &\simeq 2\pi\frac{1}{240}\left(\frac{1}{3}\delta^{-3}+\delta\right),\\
\mathrm{ln}Z_B^{(6)} &\simeq 2\pi\frac{1}{504}\left(\frac{1}{5}\delta^{-5}-\delta\right).
\end{aligned} \tag{3.6}$$

In the following we will use modular invariance to determine that the corrections to each of these formulae are exponentially small for large RT, of $\mathcal{O}(e^{-(2\pi)^2 RT})$.

As a preliminary result note that the identity

$$\zeta(s) = 2^s \pi^{s+1} \sin\frac{\pi s}{2} \Gamma(1-s)\ \zeta(1-s)\,,$$

gives the symmetry property

$$G_B^{(d)}(-s) = (-)^{d/2} G_B^{(d)}(s)\,. \tag{3.7}$$

Recalling $|B_d| = (-1)^{d/2} B_d$ for even d this is consistent with the obvious symmetry between the poles given in (3.3) and (3.4). Before proceeding we introduce the notation

$$I_B^{(d)}(\delta) = -\delta^{d/2} \frac{\partial}{\partial \delta} \ln\left[q^{-\gamma_d^B} Z_B^{(d)}\right]\,,$$

where

$$\gamma_d^B = \frac{B_d}{2d}\,, \tag{3.8}$$

generalizes the shift $c/24 = 1/24$ familiar in the CFT of a free boson in $D = 2$. Now, deform the contour C in (3.2) from large positive $\mathrm{Re}\, s$ to large negative $\mathrm{Re}\, s$ by integrating over the poles. Then use the symmetry (3.7) to return the contour to large positive $\mathrm{Re}\, s$. The result can be written

$$I_B^{(d)}\left(\frac{1}{\delta}\right) = (-1)^{d/2}\ I_B^{(d)}(\delta)\,. \tag{3.9}$$

The role of the shift (3.8) is to take the poles correctly into account. For brevity we have omitted the zero-mode for $d = 2$.

The modular relation (3.9) gives an alternative derivation of the power law terms (3.5) (and therefore (3.6)). Simply use (3.9) to relate the behavior at small δ to that of large δ; and then note that at large δ (3.1) makes it manifest that the partition function is a power series in $e^{-2\pi\delta} = e^{-(2\pi)^2 RT}$.

3.2 The fermions

The considerations above can be repeated for fermions. The basic partition function is

$$Z_F^{(d)} = \prod_{n=0}^{\infty} \left(1 + q^{n+1/2}\right)^{(n+1/2)^{d-2}}\,.$$

Carrying out the steps (3.1) through (3.2) we find the Mellin transform

$$-\delta^{d/2} \frac{\partial}{\partial \delta} \ln Z_F^{(d)} = \frac{1}{2\pi i} \int_C \delta^{-s} G_F^{(d)}(s) ds\,,$$

where

$$G_F^{(d)}(s) = G_B^{(d)}(s) \left(1 - \frac{1}{2^{s-1+d/2}}\right) \left(2^{s-d/2+1} - 1\right)\,.$$

The pole of $G_B^{(d)}$ at $s = 0$ for $d = 2$ is cancelled by the first term in brackets. This reflects the absence of zero-modes for (NS) fermions. The only poles are therefore

$$G_F^{(d)}(s) \sim 2\pi \left(1 - \frac{1}{2^{d-1}}\right) \frac{|B_d|}{2d} \frac{1}{s - d/2}\,, \qquad s \sim \frac{d}{2}\,,$$

and its mirror image under the symmetry

$$G_F^{(d)}(-s) = (-)^{d/2} G_F^{(d)}(s)\,.$$

This symmetry of $G_F^{(d)}$ is the key step in showing that modular symmetry is maintained for fermions. In other words

$$I_F^{(d)}\left(\frac{1}{\delta}\right) = (-1)^{d/2}\; I_F^{(d)}(\delta)\,, \tag{3.10}$$

where now

$$I_F^{(d)}(\delta) = -\delta^{d/2} \frac{\partial}{\partial\delta} \ln\left[q^{-\gamma_d^F} Z_F^{(d)}\right].$$

We introduced

$$\gamma_d^F = \left(1 - \frac{1}{2^{d-1}}\right) \gamma_d^B = \left(1 - \frac{1}{2^{d-1}}\right) \frac{B_d}{2d}\,, \tag{3.11}$$

generalizing the shift $c/24 = 1/48$ familiar from the CFT of a free (NS) fermion in $d = 2$.

The leading (power series) behavior of the partition functions is related to that of the bosons in a simple way

$$\ln Z_F^{(d)} \simeq \left(1 - \frac{1}{2^{d-1}}\right) \ln Z_B^{(d)}\,.$$

This is well-known for the thermodynamic behavior at large temperature; the modular invariance implies that it also holds for the Casimir corrections.

There are more general boundary conditions that could be analyzed similarly (e.g. generalizing R fermions to d dimensions). These functions do not appear in the examples considered in this paper.

4. Statistical mechanics of free fields

In this section we use the results of section 3 to analyze the partition functions of physical interest.

4.1 Four dimensions

The simplest way to analyze the partition functions (2.10) for the various fields is to decompose them into the basic ones considered in detail in the previous sections.

The result is

$$
\begin{aligned}
Z_S^{(4)} &= Z_B^{(4)}\,, \\
Z_W^{(4)} &= (Z_F^{(4)})^2\ (Z_F^{(2)})^{-1/2}\,, \\
Z_V &= (Z_B^{(4)})^2\ (Z_B^{(2)})^{-2}\,.
\end{aligned} \tag{4.1}
$$

The qualitative features of the partition functions therefore follow from those of the basic ones analyzed in section 3. Note however that the "two dimensional" partition functions also enter the results in four dimensions. Since the precise form of the modular symmetry depends on the index d it follows that the full partition functions in $D=4$ do *not* satisfy any modular symmetry, except for the boson (the case considered by Cardy [8]). This is not a problem because there is no principle requiring this kind of symmetry in dimensions larger than $D=2$ (as far as we are aware). Thus it would have been surprising if the partition functions did satisfy a modular symmetry. For our purposes the important point is that the modular symmetry *of the constituent partition functions* suffices to analyze the high temperature behavior of the partition sums (4.1). For example it ensures that the free energies are simple power series with no inverse powers of temperature, up to non-perturbative corrections.

The $\mathcal{N}=4$ SYM has the partition function

$$
Z_{N=4} = (Z_B^{(4)})^8 (Z_B^{(2)})^{-2} (Z_F^{(4)})^8 (Z_F^{(2)})^{-2}\,. \tag{4.2}
$$

For the purpose of analyzing the corresponding thermodynamics we can relate the fermionic partition function to that of the boson and find

$$
\ln Z_{N=4} \simeq 15 \ln Z_B^{(4)} - 3 \ln Z_B^{(2)} \simeq 2\pi \left[\frac{1}{16}\left(\frac{1}{3}\delta^{-3}+\delta\right) - \frac{1}{8}(\delta^{-1}-\delta)\right], \tag{4.3}
$$

up to exponential corrections. More generally, for a theory with n_S scalars, n_F Weyl fermions, and n_V Maxwell fields the polynomial part of the partition function is

$$
-FR = \frac{1}{2\pi\delta}\ln Z = a_4\delta^{-4}+a_2\delta^{-2}+a_0 = a_4(2\pi RT)^4+a_2(2\pi RT)^2+a_0(2\pi RT)^0\,, \tag{4.4}
$$

where

$$
\begin{aligned}
a_4 &= \frac{1}{720}\left(n_S+2n_V+\frac{7}{4}n_F\right), \\
a_2 &= -\frac{1}{24}\left(2n_V+\frac{1}{4}n_F\right), \\
a_0 &= \frac{1}{240}\left(n_S+22n_V+\frac{17}{4}n_F\right).
\end{aligned} \tag{4.5}
$$

These three coefficients satisfy the constraint

$$
3a_4 - a_2 - a_0 = 0\,. \tag{4.6}
$$

This relation is no surprise because it is obviously satisfied by the "constituent" partition functions (3.6). It shows that the polynomial part of the free energy in $D = 4$ in general depends on *two* independent parameters.

The generalization (1.7) of (4.6) to arbitrary dimensions follows similarly from the formulae in section 3.

4.2 Six dimensions

The six dimensional partition functions (2.11) can be similarly decomposed into the basic ones. The result is

$$\begin{aligned} Z_S^{(6)} &= (Z_B^{(6)})^{1/12}(Z_B^{(4)})^{-1/12} \,, \\ Z_W^{(6)} &= (Z_F^{(6)})^{1/3}(Z_F^{(4)})^{-5/6}(Z_F^{(2)})^{3/16} \,, \\ Z_T &= (Z_B^{(6)})^{1/4} \, (Z_B^{(4)})^{-5/4} Z_B^{(2)} \,. \end{aligned} \tag{4.7}$$

Note that here all fields, including the scalar, receive contributions from basic partition functions of the "wrong" dimension.

Combining the results for the field content of the $(2,0)$ theory gives

$$Z_{(2,0)} = (Z_B^{(6)})^{2/3}(Z_B^{(4)})^{-5/3} Z_B^{(2)} (Z_F^{(6)})^{2/3}(Z_F^{(4)})^{-5/3}(Z_F^{(2)})^{3/8} \,. \tag{4.8}$$

The leading behavior for small δ becomes

$$\begin{aligned} \ln Z_{(2,0)} &\simeq \frac{2}{3}\frac{63}{32}\ln Z_B^{(6)} - \frac{5}{3}\frac{15}{8}\ln Z_B^{(4)} + \left(1 + \frac{3}{16}\right)\ln Z_B^{(2)} \\ &\simeq 2\pi \frac{1}{12 \cdot 32}\left[\left(\frac{1}{5}\delta^{-5} - \delta\right) - 5\left(\frac{1}{3}\delta^{-3} + \delta\right) + 19(\delta^{-1} - \delta)\right], \end{aligned} \tag{4.9}$$

up to exponential corrections.

The partition functions (4.2) and (4.8) for the maximally symmetric theories share an interesting property. In each case $Z_B^{(d)}$ and $Z_F^{(d)}$ appear to the same power, as they must due to supersymmetry. The same is true in both cases for the first subleading terms $Z_B^{(d-2)}$ and $Z_F^{(d-2)}$. It is possible that this is also guaranteed by the large supersymmetry, but we have not analyzed this.

For a general theory with n_S scalars, n_F Weyl fermions, and n_T tensor fields the polynomial part of the partition function is

$$-FR = a_6\delta^{-6} + a_4\delta^{-4} + a_2\delta^{-2} + a_0 \,, \tag{4.10}$$

where

$$\begin{aligned} a_6 &= \frac{1}{2520}\left(\frac{1}{12}n_S + \frac{1}{4}n_T + \frac{31}{192}n_F\right), \\ a_4 &= -\frac{1}{720}\left(\frac{1}{12}n_S + \frac{5}{4}n_T + \frac{35}{96}n_F\right), \end{aligned}$$

$$a_2 = \frac{1}{24}\left(n_T + \frac{3}{64}n_F\right),$$
$$a_0 = -\frac{1}{4032}\left(\frac{31}{15}n_S + 191n_T + \frac{367}{24}n_F\right). \tag{4.11}$$

These four coefficients satisfy the constraint (1.7)

$$5a_6 - 3a_4 + a_2 + a_0 = 0\,.$$

Thus the polynomial part of the free energy in $D = 6$ generally depends on *three* independent parameters.

5. Entropy formulae and bounds

In this section we apply our study of free CFT's to discuss the entropy formula (1.3) recently proposed by E. Verlinde [4], and the Bekenstein bound (1.1).

5.1 An asymptotic entropy formula?

Starting from the polynomial part of the free energy (1.6) it is straightforward to work out other thermodynamic quantitites, e.g. the sub-extensive part of the energy (1.4)

$$E_C R = -\left(2a_{D-2}\delta^{-D+2} + 4a_{D-4}\delta^{-D+4} + \cdots + Da_0\delta^0\right).$$

In these manipulations we assume the thermodynamic limit where the entropy $S = \beta(E - F)$ is found by a saddle point approximation. An improved treatment would give logarithmic and higher corrections that are unimportant at high temperature.

The results of sections 2-4 imply that:

1. At large temperature the leading sub-extensive energy is governed by a_{D-2}. For general matter content (4.5) and (4.11) give $a_{D-2} \leq 0$. We therefore find $E_C \geq 0$ to the leading order. This is encouraging because a negative value of E_C renders (1.3) meaningless. However, there is a specific case where the inequality is saturated and $a_{D-2} = 0$. This is the conformally coupled scalar field in $D = 4$. Moreover, here the next order is $a_0 > 0$ and so $E_C < 0$. It is clear that some essential modification is needed in this situation.

2. Leaving aside the conformally coupled scalar in $D = 4$ we have $a_{D-2} < 0$ and so $E_C \sim (TR)^{D-2}$ for large TR. Therefore the functional dependence on TR agrees on the two sides of (1.3) for large TR. The existence of a component of the energy with $E_C \sim (TR)^{D-2}$ is a key point of the reasoning in [4]. However, although the *scaling* works correctly for large TR the *coefficients* do not in general match. For large TR, the leading term in (1.3) gives the relation

$$a_{D-2} = -\frac{D^2(D-1)}{4}\,a_D\,. \tag{5.1}$$

Remarkably this is satisfied for theories with a holographic dual [4, 10]. However, for the theories considered in this paper there is in general no relation between a_{D-2} and a_D, except for $D=2$ where $a_0=-a_2$ in agreement with (5.1). It is therefore not sufficient to modify the overall numerical coefficient of the r.h.s. of (1.3), because the change would have to depend on the matter content of the theory.

3. Finally, let us note that although the two sides of (1.3) generically have identical scaling behavior for large TR the full functional form is in general different once the subleading terms are taken into account.

In $D=4$, (5.1) can be compared with our result

$$-\frac{a_2}{a_4}=30\frac{2n_V+1/4n_F}{n_S+2n_V+7/4n_F}\,.$$

As we vary the matter content, this expression covers the *bounded* range

$$0\leq\left|\frac{a_2}{a_4}\right|\leq 30\,.$$

The bounds are saturated for pure scalar and vector theories, respectively. The corresponding range for $D=6$

$$\frac{7}{2}\leq\left|\frac{a_4}{a_6}\right|\leq\frac{35}{2}\,,$$

is saturated for scalar and tensor fields, respectively. Now let us assume that (5.1) is really a prediction for the strongly coupled theory. Then the bounds above are consistent with the notion that both the extensive a_D *and* the sub-extensive a_{D-2} coefficients flow modestly between weak and strong coupling. The physics of these coefficients may therefore be analogous to that underlying the (in)famous 3/4 factor for the entropy [11].

As an example, for $\mathcal{N}=4$ SYM in $D=4$, at *strong* coupling the formula (1.3) is valid and in particular $a_2=-12a_4$ (according to (5.1)). At *weak* coupling we find $a_2=-6a_4$. The leading coefficient in the thermodynamics depends on the coupling so that [11]

$$a_4(g^2N=\infty)=\frac{3}{4}a_4(g^2N=0)\,. \tag{5.2}$$

The corresponding relation for the leading sub-extensive part

$$a_2(g^2N=\infty)=\frac{3}{2}a_2(g^2N=0)\,.$$

When cast in this light the numerical differences between strong and weak coupling are less significant. Indeed, it is perhaps surprising that a_2 (as a_4) is so similar in the two regimes, having in particular the same overall dependence on the number of species (N^2 for SU(N) theory).

These comments were for the leading terms; the full functional dependence on RT depends dramatically on the coupling. At strong coupling the high temperature expansion of the free energy includes an *infinite* series in $1/RT$ [10, 4]. As the coupling g^2N goes to zero, the coefficients of all the terms that go like negative powers of RT go to zero and one recovers the polynomial (1.6). In particular, the relation (1.3) is far from being satisfied at small g^2N.

5.2 The role of finite size effects in violating Bekenstein's bound

In the high energy limit $ER \gg 1$ the entropy of a CFT on (1.2) is dominated by the leading term in (1.6). It can be written as

$$S \sim a_D^{1/D}(ER)^{(D-1)/D}\,. \tag{5.3}$$

The Bekenstein bound (1.1) is not very stringent at high energies, since the actual entropy (5.3) increases slower with energy than does the bound. Now, the numerical coefficient a_D is essentially the central charge c, or the number of degrees of freedom of the system (see e.g. [12] for a recent discussion). In particular, it is proportional to the number of fields in the non-interacting theory (see (4.5), (4.11)). By comparing (1.1) and (5.3) we see that the bound appears to break down for

$$ER \leq a_D\,. \tag{5.4}$$

The fact that potential violations of the bound are associated with low energies is well known [3]. It is usually phrased as the statement that the bound appears to be violated at low temperatures. What is interesting in (5.4) is that the dependence on a_D is precisely such that the relevant energy is the same as the energy at which finite size effects become important. Equivalently, the temperature at which the bound is violated is $T \sim 1/R$, independently of a_D. This leads to a number of consequences:

1. In order to determine whether or not the bound (1.1) is in fact violated in the regime (5.4) one has to consider subleading corrections to the asymptotic formula (5.3) which correspond to sub-extensive terms in the free energy. This has been carried out in detail in this paper for weakly coupled CFT's.

2. The corrections in question are not just the powerlike corrections in (1.6), but also the exponential ones. Consider for example the case of two dimensional CFT. Cardy's formula (1.5) which is obtained by considering the free energy (1.6) and ignoring the exponential corrections appears to predict that the entropy is bounded from above $S \leq 2\pi L_0$, but when the bound is close to being saturated, the temperature T is of order $1/R$. For such temperatures the exponential corrections in (1.6) are important. Similarly in higher dimensions the exponential terms are important when the bound (1.1) is close to being violated at $RT \sim 1$. In fact, the notion of temperature is not really well defined in this regime, and it is much better to work in the microcanonical ensemble and count states.

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3. In the free field examples considered in this paper it is in fact straightforward to violate the Bekenstein bound (5.4). The reason is that if one decreases the energy to $ER \sim 1$, the explicit counting shows that the entropy goes like $S \sim \log a_D$ and the bound is badly violated for a large number of species $a_D \gg 1$ [2]. The precise energy at which the bound is first violated depends on the precise definition of the bound (1.1) (which as mentioned in the introduction is unclear), but with any reasonable definition it seems natural that the violation will occur around the energy (5.4). It should be emphasized that while we have not checked this assertion, it is possible to do so by using the exact formulae for the partition sums presented in section 2, by studying the n dependence of the coefficients of q^n.

4. One might be confused at this point how the bound (1.1) could be satisfied in the strong coupling region, where the AdS calculation of [10, 4] seems to imply that it is. An example from two dimensional CFT might be useful in this regard. Consider a sigma model with target space T^{4N} as a toy model of a "weakly coupled" CFT in the higher dimensional setup, and CFT on T^{4N}/S_N as an example of a "strongly coupled" CFT. In the T^{4N} case, the free energy is just N times that of CFT on T^4 and the arguments of point (3) above apply; one expects the bound to be violated at L_0 of order N. For T^{4N}/S_N there are very few states with L_0 of order one, and the arguments of point (3) do not imply that the bound must be violated. In fact according to [10, 4] it is not violated in this case.

Regardless of what one finds in any particular model, it seems that the bound (1.1) is unlikely to apply in general in the regime (5.4) since while the high energy behavior (5.3) is universal, the experience in CFT is that the growth in the number of states for low and intermediate energies is quite model dependent. It is thus natural to propose that the bound (1.1) only applies in the thermodynamic limit, when finite size effects are negligible. Such a modification would not be acceptable if the bound (1.1) was needed for the validity of the GSL of thermodynamics, but if it is not needed (as argued e.g. in [2]) we see nothing wrong with it.

Finally, in [3] it was argued that zero point energy might be crucial in restoring the bound (1.1) even when it naively seems to break down. In our context this does not seem to be the case. It is true that in computing the free energy we have assigned zero energy to the vacuum on (1.2). For free fields, there is no mystery in the zero point energy – it is simply the contribution of the free fields to the cosmological constant. For example, it cancels between bosons and fermions in all the supersymmetric cases mentioned above. It should also be noted, that the AdS calculations that establish the bound for strongly coupled CFT's [10, 4], assign zero energy to the vacuum on S^{D-1}.

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